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KEYWORDS

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When the auctioneer (which is the buyer) can commit ex ante to the slope of demand, the equilibrium is a Pareto improvement relative to the simultaneous benchmark.
This is because the sellers' slope choice generates positive externalities on the buyer, so when committing, the buyer has incentives to choose a steeper slope.
As a result, commitment mitigates the supply reduction due to market power: the equilibrium has both a higher price and higher quantity, so that both the buyer and the sellers are strictly better off.

Optimal Linear Demand in a Uniform-Price Procurement Auction*

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April 1, 2026

Abstract

This note studies the optimal demand commitment in a uniform-price procurement auction, within the class of linear schedules. When the auctioneer (which is the buyer) can commit ex ante to the slope of demand, the equilibrium is a Pareto improvement relative to the simultaneous benchmark. This is because the sellers' slope choice generates positive externalities on the buyer, so when committing, the buyer has incentives to choose a steeper slope. As a result, commitment mitigates the supply reduction due to market power: the equilibrium has both a higher price and higher quantity, so that both the buyer and the sellers are strictly better off.

Keywords: Uniform-price auction, auction design, supply function equilibrium, procurement
JEL Classification: C72, D43, D44, D47, H57

1 Introduction

How should a buyer design demand in a uniform-price procurement auction when sellers respond strategically? This question is natural in applications in which

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a centralized buyer procures a homogeneous input from multiple suppliers. In the standard uniform-price auction framework, the buyer's demand is typically treated as exogenous, while strategic behavior is studied on the supply side. In many procurement settings, however, the buyer can influence market outcomes through the shape of the demand schedule that it commits to ex ante. This note studies that problem.

I consider a uniform-price procurement auction in which a buyer commits ex-ante to a linear demand schedule and sellers submit supply schedules. Sellers have quadratic costs, and the market clears at the price at which aggregate supply equals demand (the auction is a uniform-price auction). The game among sellers for a given demand chosen by the buyer is exactly the game studied in Klemperer and Meyer (1989). In particular, they show that for a linear demand, the linear equilibrium is unique, provided the uncertainty over the demand intercept is large enough. Moreover, the reduced game where agents just decide the slopes is a supermodular game, which simplifies considerably the comparative statics. Because of these reasons, I study the optimal choice of the buyer among *linear* demand functions.

The auctioneer has a linear-quadratic payoff with decreasing marginal utility from additional units. Lemma 4.1 characterizes the optimal demand slope that maximizes the auctioneer payoff. The main result is that commitment to a linear demand slope is Pareto improving with respect to the non-commitment case. In the no-commitment case, the auctioneer and the sellers submit their schedules simultaneously: this can be interpreted as a situation in which the auctioneer can adjust ex-post the quantity demanded, but does not commit ex-ante to a price-quantity schedule.

In the simultaneous game, the buyer chooses its slope taking seller behavior as given. With commitment, instead, the buyer internalizes the effect of its slope on the sellers' subsequent supply responses. By strategic complementarity, a steeper demand schedule induces sellers to choose steeper supply schedules. Crucially, steeper supply schedules tend to decrease the price, so they represent a positive externality on the buyer: this incentivizes the buyer to choose a steeper slope with commitment. By strategic complementarities, this comparison remains true in equilibrium. From an economic point of view, commitment reduces the market power inefficiency: each quantity becomes marginally cheaper for the buyer, so the buyer ends up buying more units, in such a way that also makes sellers better off.

Related literature The paper relates to two strands of literature. First, it contributes to the literature on the design of uniform-price auctions. Kremer and Nyborg (2004) and LiCalzi and Pavan (2005) study the optimal design of a supply schedule (the auctioneer is a seller in these papers, rather than a buyer) with the goal of dealing with the multiplicity of underpricing equilibria. The present paper, instead, focuses on the unique equilibrium of the supply function competition among sellers under uncertainty, identified by Klemperer and Meyer (1989), analyzing the optimal demand choice by the auctioneer. Back and Zender (1993) and McAdams (2007) find that the auctioneer can profit from non-commitment; here we show that commitment is Pareto-improving. Ausubel et al. (2014) show that uniform and discriminatory price auctions are not strictly ranked by revenues, which one is optimal depends on the setting. Baldwin et al. (2024) consider more complex multi-unit auction environments that can implement a competitive equilibrium for a given reported schedule of the auctioneer, but do not study the optimal demand.

Second, it contributes to the literature on supply function equilibria. The main difference with respect to Klemperer and Meyer (1989), Green and Newbery (1992) and, more recently, Holmberg et al. (2025), is that I study the dynamic version in which one player (the auctioneer) commits to its schedule before the other players. Other papers study design of financial markets where traders use similar uniform price auctions: Rostek and Yoon (2021b), Rostek and Yoon (2021a) and Chen (2024). All these papers focus on the design of double auctions as a trading platform or market, and do not study procurement design.

2 Setting

In this section, we study the optimal schedule for an auctioneer designing a multiunit uniform price auction.

Consider a procurement auction in which the buyer/auctioneer wants to buy a quantity Q of the good in order to optimize the payoff:

$$U_b = \left(\varepsilon Q - \frac{1}{2} Q^2 - pQ \right)$$

There are $N \geq 2$ sellers with cost function $\frac{1}{2k} q_i^2$. The auction format is a uniform price auction. So, agents must submit bids (schedules) $q_i(p)$, and the auctioneer commits ex-ante to a demand function $D(p)$. The price is the unique one that solves

$$D(p) = \sum_i q_i(p)$$

We restrict attention to the schedules of the form: $q_i(p) = \overline{B}_i p$ and $D(p) = B_c(\varepsilon - p)$, for $\overline{B}_i > 0$ and $B_c > 0$. Under these conditions, the price is unique and equal to:

$$p = \frac{\varepsilon B_c}{B_c + \sum_j \overline{B}_j} \tag{1}$$

So, if the equilibrium price is p, the payoff of sellers is:

$$U_i = pq_i(p) - \frac{1}{2k} q_i^2(p)$$

We explore in detail two games:

Game without commitment G is the game in which auctioneer and sellers choose the slopes *simultaneously*;

Game with commitment is the game G^{com} in which the auctioneer moves first, committing to the slope B_c; the sellers observe such a slope then, simultaneously, choose the supply slopes.

3 Simultaneous benchmark (no commitment)

If the auctioneer had no commitment, the auctioneer would simply behave as a player in the market, as in the trading games of Malamud and Rostek (2017) or Chen (2024). In the benchmark setting of Klemperer and Meyer (1989) the auctioneer has a fixed demand function. The following lemma adapts the reasoning to this case.

Lemma 3.1. In the simultaneous model, there is a unique equilibrium where the seller slope is homogeneous $\overline{B}_i = \overline{B}$, and the slopes are the solution of:

$$B_c = \left(1 + \frac{1}{N\overline{B}}\right)^{-1} \tag{2}$$

$$\overline{B} = \left(\frac{1}{k} + \frac{1}{B_c + (N-1)\overline{B}}\right)^{-1} \tag{3}$$

Moreover, in this equilibrium we have that $B_c \in [0, 1]$ and $\overline{B} \in [0, k]$.

Remark 3.1. In the simultaneous model of Klemperer and Meyer (1989), the linear equilibrium is an equilibrium of the game in which sellers can choose any nonlinear schedule. Moreover, it is the unique equilibrium in a modified model in which the buyer intercept is stochastic. However, in the sequential model the optimal schedule chosen by the buyer is not necessarily linear. The analysis of the fully unrestricted optimal schedule requires solving a nonlinear system of ordinary differential equations, so it is much less tractable. For this reason, we study the optimal *linear* demand function, restricting the auctioneer in the sequential model to choose a function in the same class as the equilibrium without commitment: $D(p) = B_c(\varepsilon - p)$ for some real slope B_c.

4 Optimal linear demand function

The goal of the section is to characterize the equilibrium with buyer commitment and compare it with the simultaneous benchmark.

The optimal choice of the auctioneer is characterized by an implicit equation, in the following Lemma.

Lemma 4.1. The model with commitment has a unique linear equilibrium, and the equilibrium slopes are pinned down by:

$$\overline{B} = \left(\frac{1}{k} + \frac{1}{B_c + (N-1)\overline{B}}\right)^{-1} \tag{4}$$

$$B_c = \left(1 + \frac{1}{N\overline{B}} - \frac{(2 - B_c)}{N\overline{B}^2} B_c \frac{d\overline{B}}{dB_c}\right)^{-1} \tag{5}$$

Theorem 1. *1. In the unique equilibrium of the uniform price auction with commitment the slopes are higher than without commitment.*

2. As a consequence, the price is higher, and the equilibrium profits are higher for both the buyer and all the sellers:

$$p^{seq} > p^{\text{sim}}, \qquad U_b^{seq} > U_b^{\text{sim}}, \qquad U_i^{seq} > U_i^{\text{sim}} \quad \textit{for all } i = 1, \dots, N.$$

Part 1 is proved directly from the monotonicity of the sellers' equilibrium reaction and the buyer's reduced objective. For Part 2, the key fact is that the price depends on the relative equilibrium slopes of buyers and sellers:

$$p = \frac{\varepsilon}{1 + \frac{N\overline{B}}{B_c}}$$

As a consequence, the key object is the ratio of slopes. The key intuition is that the reaction of sellers is always less than one-to-one for any increase in the slope of the buyer: as a consequence, $\frac{N\overline{B}}{B_c}$ is decreasing in B_c, and so the price is increasing in B_c. It is clear why sellers benefit from a higher slope B_c: it means both a higher quantity sold, and higher prices. The nontrivial part is that *also the buyer* benefits: this is because a higher slope of sellers has positive externalities on the buyer, because it decreases the price. So, the buyer trades off higher prices and higher quantity bought, but the positive externalities mean that the buyer resolves this trade-off in the direction of a higher slope. The details are in Appendix B.2.

5 Conclusion

Uniform-price auctions are a widely used form of multi-unit auctions. We make progress towards characterizing the optimal design of uniform price auctions in a procurement context, showing that commitment can generate Pareto gains for all agents involved.

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Appendix

A Proof of Section 3

A.1 Proof of Lemma 3.1

Substituting the linear schedules into the payoffs, we obtain:

$$U_b^{sim} := \left(B_c - \frac{1}{2}B_c^2\right)(\varepsilon - p)^2 \tag{6}$$

$$U_i^{sim} := \left(\overline{B}_i - \frac{1}{2k}\overline{B}_i^2\right)p^2 \tag{7}$$

where the price is the one defined in (1).

Taking the first-order condition, we obtain that the best reply of sellers is, as in Klemperer and Meyer (1989):

$$\overline{B}_i = \left(\frac{1}{k} + \frac{1}{B_c + \sum_{j\neq i}\overline{B}_j}\right)^{-1}$$

By Klemperer and Meyer (1989) results, the best reply of the sellers is unique, all sellers have the same slope $\overline{B}_i = \overline{B}$, and this satisfies:

$$\overline{B} = \left(\frac{1}{k} + \frac{1}{B_c + (N-1)\overline{B}}\right)^{-1}$$

So, we can limit ourselves to a setting where sellers choose the same slope $\overline{B}$. Given the slopes of sellers, choosing a quantity is equivalent to choosing a linear slope. So, the best reply problem of the buyer can be rewritten as:

$$\max_{Q} \varepsilon Q - \frac{1}{2}Q^2 - pQ$$
$$\text{s.t. } Q = N\overline{B}p$$

which gives:

$$\varepsilon - Q - Q/(N\overline{B}) - p = 0$$
$$Q = \left(1 + \frac{1}{N\overline{B}}\right)^{-1}(\varepsilon - p)$$

So, it follows that the auctioneer demand is also linear:

$$D(p) = \left(1 + \frac{1}{N\overline{B}}\right)^{-1}(\varepsilon - p)$$

with slope $B_c = \left(1 + \frac{1}{N\overline{B}}\right)^{-1}$ and intercept $A = \varepsilon B_c$. This pins down the equilibrium equations (3) in the text.

Now, we analyze the equilibrium of a modified game G^W, with payoffs defined as follows:

$$W_b := \left(B_c - \frac{1}{2}B_c^2\right)p^2 \tag{8}$$

$$W_i := \left(\overline{B} - \frac{1}{2k}\overline{B}^2\right)p^2 \tag{9}$$

and:

$$p^w = \frac{\varepsilon}{N\overline{B} + B_c + C}$$

This game is a special case of Bizzarri (2022). In that paper, it is show that this game has a unique linear equilibrium, for any $C \geq 0$. The equilibrium equations are:

$$B_c = \left(1 + \frac{1}{C + N\overline{B}}\right)^{-1}$$

$$\overline{B} = \left(\frac{1}{k} + \frac{1}{C + B_c + (N-1)\overline{B}}\right)^{-1}$$

We can see that, taking $C = 0$, they reduce exactly to the equilibrium equations of our simultaneous game 3: we conclude that such a game has a unique equilibrium.

Finally, since the slopes are assumed nonnegative, the equilibrium equations (3) immediately imply $B_c \leq 1$ and $\overline{B} \leq k$.

B Proof of Section 4

B.1 Proof of Lemma 4.1

In the second stage the equilibrium is as in Klemperer and Meyer (1989), so it is homogeneous $\overline{B}_i = \overline{B}$, unique, and pinned down by the equation:

$$\overline{B} = \left(\frac{1}{k} + \frac{1}{B_c + (N-1)\overline{B}}\right)^{-1},$$

The buyer internalizes this in the first stage.

By Lemmas B.2 and B.3, maximizing the buyer payoff over B_c is equivalent to maximizing $\widetilde{U}_b(S)$ over $S \in \mathcal{I}_S$, and so there is a unique maximizer satisfying the FOC. As a consequence, the SPE is unique.

The effect of B_c is:

$$\frac{d\overline{B}}{dB_c} = -\frac{\left(\frac{1}{k}+\frac{1}{B_c+(N-1)\overline{B}}\right)^{-2}\frac{1}{\left(B_c+(N-1)\overline{B}\right)^2}}{\left(\frac{1}{k}+\frac{1}{B_c+(N-1)\overline{B}}\right)^{-2}\frac{N-1}{\left(B_c+(N-1)\overline{B}\right)^2}-1} \tag{10}$$

$$= -\frac{\frac{\overline{B}^2}{\left(B_c+(N-1)\overline{B}\right)^2}}{\frac{(N-1)\overline{B}^2}{\left(B_c+(N-1)\overline{B}\right)^2}-1} = \frac{\frac{\overline{B}^2}{\left(B_c+(N-1)\overline{B}\right)^2}}{1-\frac{(N-1)\overline{B}^2}{\left(B_c+(N-1)\overline{B}\right)^2}} \tag{11}$$

$$= \frac{\overline{B}^2}{\left(B_c+(N-1)\overline{B}\right)^2-(N-1)\overline{B}^2} > 0 \tag{12}$$

The payoff of the buyer in the first stage is:

$$U_b = \varepsilon B_c(\varepsilon-p) - \frac{1}{2}B_c^2(\varepsilon-p)^2 - pB_c(\varepsilon-p) = B_c\left(1-\frac{B_c}{2}\right)(\varepsilon-p)^2$$

The effect on the price is:

$$\frac{dp}{dB_c} = \varepsilon\frac{\left(B_c+N\overline{B}\right)-B_c\left(1+N\frac{d\overline{B}}{dB_c}\right)}{\left(B_c+N\overline{B}\right)^2} = \varepsilon\frac{N\left(\overline{B}-B_c\frac{d\overline{B}}{dB_c}\right)}{\left(B_c+N\overline{B}\right)^2} \tag{13}$$

$$= \varepsilon\frac{N\overline{B}}{\left(B_c+N\overline{B}\right)^2}\left(1-\frac{d\overline{B}}{dB_c}\frac{B_c}{\overline{B}}\right) \tag{14}$$

$$= \frac{(\varepsilon-p)}{\left(B_c+N\overline{B}\right)}\left(1-\frac{d\overline{B}}{dB_c}\frac{B_c}{\overline{B}}\right) \tag{15}$$

Moreover:

$$1-\frac{d\overline{B}}{dB_c}\frac{B_c}{\overline{B}} = 1-\frac{\overline{B}^2}{\left(B_c+(N-1)\overline{B}\right)^2-(N-1)\overline{B}^2}\frac{B_c}{\overline{B}} \tag{16}$$

$$= \frac{\left(B_c+(N-1)\overline{B}\right)^2-(N-1)\overline{B}^2-\overline{B}B_c}{\left(B_c+(N-1)\overline{B}\right)^2-(N-1)\overline{B}^2} \tag{17}$$

$$= \frac{B_c^2+(2N-3)B_c\overline{B}+(N-1)(N-2)\overline{B}^2}{\left(B_c+(N-1)\overline{B}\right)^2-(N-1)\overline{B}^2} > 0 \tag{18}$$

So the price increases with B_c, as expected (market power of buyer corresponds to lower B_c). The FOC are:

$$(1-B_c)(\varepsilon-p)^2 - 2B_c\left(1-\frac{B_c}{2}\right)(\varepsilon-p)\left(\frac{(\varepsilon-p)}{\left(B_c+N\overline{B}\right)} - \frac{\varepsilon N B_c \frac{d\overline{B}}{dB_c}}{\left(B_c+N\overline{B}\right)^2}\right) = 0 \tag{19}$$

$$(1-B_c)(\varepsilon-p)^2 - 2B_c\left(1-\frac{B_c}{2}\right)\frac{(\varepsilon-p)^2}{\left(B_c+N\overline{B}\right)}\left(1-\frac{d\overline{B}}{dB_c}\frac{B_c}{\overline{B}}\right) = 0 \tag{20}$$

$$(1-B_c)\left(B_c+N\overline{B}\right) - B_c(2-B_c)\left(1-\frac{d\overline{B}}{dB_c}\frac{B_c}{\overline{B}}\right) = 0 \tag{21}$$

$$B_c - B_c^2 + N\overline{B}(1-B_c) - B_c(2-B_c) + B_c(2-B_c)\frac{d\overline{B}}{dB_c}\frac{B_c}{\overline{B}} = 0 \tag{22}$$

$$-B_c + N\overline{B}(1-B_c) + (2-B_c)B_c^2\frac{d\overline{B}}{dB_c}/\overline{B} = 0 \tag{23}$$

$$\frac{N\overline{B}}{\left(1+N\overline{B}-(2-B_c)\frac{d\overline{B}}{dB_c}\frac{B_c}{\overline{B}}\right)} = B_c \tag{24}$$

So, the best reply is the unique value solving:

$$B_c = \left(1+\frac{1}{N\overline{B}}\left(1-(2-B_c)\frac{d\overline{B}}{dB_c}\frac{B_c}{\overline{B}}\right)\right)^{-1}$$

B.2 Proof of Theorem 1

In the proof, we need the following Lemmas.

Lemma B.1. Define the second stage best reply as $\beta(B_c) := \overline{B}$, where $\overline{B}$ is the unique positive solution of:

$$\overline{B} = \left(\frac{1}{k}+\frac{1}{B_c+(N-1)\overline{B}}\right)^{-1}$$

Then:

$$\beta(B_c) = \frac{(N-2)k - B_c + \sqrt{\left((N-2)k-B_c\right)^2 + 4(N-1)B_c k}}{2(N-1)}.$$

The function β is strictly increasing in B_c on $(0,\infty)$.

Lemma B.2 (Reparameterization by the aggregate seller slope)**.** Let

$$S(B_c) := \sum_{i=1}^{N} \overline{B}_i(B_c) = N\overline{B}(B_c),$$

where $\overline{B}(B_c)$ is the unique symmetric second-stage equilibrium slope. Then $S(B_c)$ is strictly increasing on $(0, +\infty)$, hence invertible on its image and:

$$B_c(S) = \frac{S\big((N-1)S - Nk(N-2)\big)}{N(Nk - S)}.$$

The admissible range of S corresponding to $B_c \in [0, +\infty)$ is

$$\mathcal{I}_S = \begin{cases} (0,\, 2k), & N = 2, \\ \left[\dfrac{Nk(N-2)}{N-1},\, Nk\right), & N > 2. \end{cases}$$

Therefore the buyer payoff along the follower equilibrium can be reparameterized as

$$\widetilde{U}_b(S) := U_b(B_c(S), S), \qquad S \in \mathcal{I}_S,$$

and maximizing $U_b(B_c, S(B_c))$ over $B_c \in [0, +\infty)$ is equivalent to maximizing $\widetilde{U}_b(S)$ over $S \in \mathcal{I}_S$.

Lemma B.3 (Single-peakedness of the reparameterized buyer payoff)**.** Consider the reparameterized payoff $\widetilde{U}_b$ of Lemma B.2. The function $\widetilde{U}_b$ has a unique global maximizer on $\mathcal{I}_S$, which is the unique point in which $\widetilde{U}_b'(S) = 0$.

Part 1 Fix a buyer slope $B_c \geq 0$. Denote the slope in the unique symmetric second-stage equilibrium by $\beta(B_c) = \overline{B}(B_c)$. By Lemma B.1, the equilibrium reaction β is increasing in B_c.

Now define the buyer's reduced objective in the sequential game:

$$V(B_c) := U_b\big(B_c, \beta(B_c)\big).$$

By Lemma 4.1, this objective has a unique maximizer; denote it by B_c^{seq}.

Let B_c^{sim} denote the buyer slope in the simultaneous equilibrium. Since in the simultaneous game the buyer takes sellers' slopes as fixed, B_c^{sim} satisfies

$$\frac{\partial U_b}{\partial B_c}\Big(B_c^{sim}, \beta(B_c^{sim})\Big) = 0.$$

Differentiating the reduced objective V gives

$$V'(B_c) = \frac{\partial U_b}{\partial B_c}\big(B_c, \beta(B_c)\big) + \sum_{i=1}^{N} \frac{\partial U_b}{\partial \overline{B}_i}\big(B_c, \beta(B_c)\big)\, \beta_i'(B_c).$$

Evaluating at B_c^{sim} and using the simultaneous first-order condition,

$$V'(B_c^{sim}) = \sum_{i=1}^{N} \frac{\partial U_b}{\partial \overline{B}_i}\Big(B_c^{sim}, \beta(B_c^{sim})\Big)\, \beta_i'(B_c^{sim}).$$

In the auction, a higher seller slope reduces the price impact of procurement and is therefore beneficial to the buyer. The derivative, evaluated at the symmetric profile, is:

$$\frac{\partial U_b}{\partial \overline{B}_i} = -2\left(B_c - \frac{1}{2}B_c^2\right)(\varepsilon - p)\frac{\partial p}{\partial \overline{B}_i} \tag{25}$$

$$= -2\left(B_c - \frac{1}{2}B_c^2\right)(\varepsilon - p)\left(-\frac{\varepsilon B_c}{(B_c + N\overline{B})^2}\right) > 0 \tag{26}$$

Moreover, $\beta_i'(B_c) > 0$ for all i by Lemma B.1. Therefore

$$V'(B_c^{sim}) > 0.$$

By Lemma B.3, since $S(B_c)$ is strictly increasing, V is single-peaked, and so it follows that

$$B_c^{seq} > B_c^{sim},$$

and since $\overline{B}(B_c)$ is increasing in B_c,

$$\overline{B}^{seq} := \overline{B}(B_c^{seq}) > \overline{B}(B_c^{sim}) := \overline{B}^{sim}.$$

Hence both the buyer slope and the sellers' equilibrium slope are higher with commitment.

Part 2 The equilibrium price is:

$$p = \frac{\varepsilon}{1 + N\overline{B}/B_c}.$$

Now we show that the relative slope $\overline{B}/B_c$ is decreasing. We have:

$$\frac{d\overline{B}}{dB_c}\frac{1}{B_c} - \frac{\overline{B}}{B_c^2} < 0 \iff \frac{d\overline{B}}{dB_c}\frac{B_c}{\overline{B}} < 1.$$

Using the expression for the derivative implied by the sellers' equilibrium equation,

$$\frac{d\overline{B}}{dB_c}\frac{B_c}{\overline{B}} = \frac{\overline{B}B_c}{B_c^2 + 2(N-1)B_c\overline{B} + (N-1)(N-2)\overline{B}^2} < 1 \tag{27}$$

if $N \geq 2$.

The buyer payoff is higher with commitment, because the value of the slope in the simultaneous equilibrium B_c^{sim} remains feasible in the sequential model, and the unique optimal value in the sequential model satisfies $B_c^{seq} > B_c^{sim}$: so, the payoff of the auctioneer must be strictly higher:

$$V(B_c^{seq}) > V(B_c^{sim}).$$

Since $\overline{B} \leq k$, seller i's payoff is increasing both in the own slope $\overline{B}$ and in the price, so they have a higher payoff $U^{seq} > U^{sim}$.

So, the equilibrium with commitment is Pareto improving with respect to the equilibrium without commitment.

B.3 Proof of Lemma B.1

Write

$$\beta_i(B_c) = \frac{(N-2)k - B_c + \sqrt{D(B_c)}}{2(N-1)}, \qquad D(B_c) := B_c^2 + 2NkB_c + (N-2)^2k^2.$$

Differentiating,

$$\beta_i'(B_c) = \frac{1}{2(N-1)}\left(-1 + \frac{B_c + Nk}{\sqrt{D(B_c)}}\right).$$

Now

$$(B_c + Nk)^2 - D(B_c) = 4(N-1)k^2 > 0,$$

so $B_c + Nk > \sqrt{D(B_c)}$. Therefore $\beta_i'(B_c) > 0$ for every $B_c > 0$.

B.4 Proof of Lemma B.2

In the homogeneous one-layer model, the followers' equilibrium condition is

$$\overline{B} = \left(\frac{1}{k} + \frac{1}{B_c + (N-1)\overline{B}} \right)^{-1}.$$

Multiplying through and using $S = N\overline{B}$ gives

$$(N-1)S^2 + NS\big(B_c - k(N-2)\big) - N^2 k B_c = 0.$$

Solving for B_c yields

$$B_c(S) = \frac{S\big((N-1)S - Nk(N-2)\big)}{N(Nk - S)}.$$

By Lemma B.1, $\overline{B}'(B_c) > 0$, hence

$$S'(B_c) = N\overline{B}'(B_c) > 0.$$

Therefore $S(B_c)$ is strictly increasing and so invertible on its image.

It remains to identify the image of $(0, +\infty)$.

If $N = 2$, then

$$B_c(S) = \frac{S^2}{2(2k - S)},$$

so $B_c(S) \geq 0$ if and only if $S \in [0, 2k)$.

If $N > 2$, then

$$B_c(S) = \frac{S\big((N-1)S - Nk(N-2)\big)}{N(Nk - S)}.$$

Since $S \geq 0$, we have $B_c(S) \geq 0$ if and only if numerator and denominator have the same sign, namely

$$\frac{Nk(N-2)}{N-1} \leq S < Nk.$$

Moreover,

$$B_c\left(\frac{Nk(N-2)}{N-1} \right) = 0, \qquad \lim_{S \uparrow Nk} B_c(S) = +\infty.$$

Thus the admissible range is exactly

$$\mathcal{I}_S = \begin{cases} (0,\, 2k), & N = 2, \\ \left[\dfrac{Nk(N-2)}{N-1},\, Nk \right), & N > 2. \end{cases}$$

Finally, since $S \mapsto B_c(S)$ is strictly increasing on $\mathcal{I}_S$, the change of variable preserves the maximizer:

$$\arg\max_{B_c \geq 0} U_b(B_c, S(B_c)) = B_c\left(\arg\max_{S \in \mathcal{I}_S} \widetilde{U}_b(S) \right).$$

This proves the claim.

B.5 Proof of Lemma B.3

Substituting

$$B_c(S) = \frac{S\big((N-1)S - Nk(N-2)\big)}{N(Nk - S)}$$

into

$$U_b(B_c, S) = \varepsilon^2 \left(B_c - \frac{1}{2} B_c^2 \right) \frac{S^2}{(B_c + S)^2},$$

and differentiating with respect to S, one obtains

$$\widetilde{U}_b'(S) = \varepsilon^2 \, \frac{\Psi(S)}{(2Nk - S)^3},$$

with

$$\begin{aligned} \Psi = {} & (N-1)^2 S^4 - N(N-1)(5Nk - 6k - 1)S^3 + 6N^2 k(N-1)(Nk - 2k - 1)S^2 \\ & - N^3 k^2 (2N^2 k - 8Nk - 7N + 8k + 10)S - 2N^4 k^3 (N-2). \end{aligned}$$

Since $S < Nk$ on $\mathcal{I}_S$, the denominator $(2Nk - S)^3$ is strictly positive on $\mathcal{I}_S$. Therefore the sign of $\widetilde{U}_b'(S)$ is exactly the sign of $\Psi(S)$.

A direct computation gives

$$\Psi''(S) = -6(N-1)(2Nk - S)\Big(2(N-1)S + N\big(1 - k(N-2)\big) \Big).$$

Now, for every $S \in \mathcal{I}_S$, we have $2Nk - S > 0$. If $N > 2$, then $S \geq \dfrac{Nk(N-2)}{N-1}$, hence

$$2(N-1)S+N\big(1-k(N-2)\big) \geq 2Nk(N-2)+N\big(1-k(N-2)\big) = N\big(k(N-2)+1\big) > 0,$$

so $\Psi''(S) < 0$. If $N = 2$, we have $2(N-1)S + N(1-k(N-2)) = 2S+2 > 0$ and again follows $\Psi''(S) < 0$. Therefore Ψ is strictly concave on $\mathcal{I}_S$.

We now evaluate the endpoints. If $N > 2$, let

$$S_0 = \frac{Nk(N-2)}{N-1}.$$

Then

$$\Psi(S_0) = \frac{N^5k^3(N-2)}{(N-1)^2} > 0, \qquad \Psi(Nk) = -N^4k^3(Nk+k+1) < 0.$$

If $N = 2$, then $\Psi(Nk) = \Psi(2k) < 0$, but $\Psi(S_0) = \Psi(0) = 0$, so to conclude that Ψ is positive close to zero we have to evaluate the derivative. Since $\Psi'(0) = 32k^2 > 0$, we conclude that, in either case, Ψ is positive at the left end of the admissible interval (or immediately to its right when $N = 2$) and negative near the right end.

Since Ψ is strictly concave, the upper contour set of zero must be convex. Since it is positive at the left end of the interval, there can be only one zero, otherwise the upper contour set would not be convex. Hence, $\widetilde{U}_b'(S)$ changes sign exactly once, from positive to negative. Therefore $\widetilde{U}_b$ is single-peaked and has a unique global maximizer on $\mathcal{I}_S$.